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# Some gap power series in multidimensional setting

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Abstract. We study extensions of classical theorems on gap power series of a complex variable to the multidimensional case.

## 1. Power series with Ostrowski gaps.

Let

$$f(z) = \sum_{0}^{\infty} Q_j(z), \quad \text{where} \quad Q_j(z) = \sum_{|\alpha|=j} a_\alpha z^\alpha, \quad \alpha \in \mathbb{Z}_+^N, \tag{1.1}$$

be a *power series* in $\mathbb{C}^N$, i.e. a series of homogeneous polynomials $Q_j$ of $N$ complex variables of degree $j$.

The set $\mathcal{D}$ given by the formula $\mathcal{D} := \{a \in \mathbb{C}^N;$ the sequence (1.1) is convergent in a neighborhood of $a\}$ is called a *domain of convergence* of (1.1).

It is known that

$$\mathcal{D} = \{z \in \mathbb{C}^N; \psi^*(z) < 1\}, \tag{1.2}$$

where

$$\psi(z) := \limsup_{j\to\infty} \sqrt[j]{|Q_j(z)|}, \tag{1.3}$$

and $\psi^*$ denotes the upper semicontinuous regularization of $\psi$.

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If $\psi^*$ is finite, then it is plurisubharmonic and absolutely homogeneous (i.e. $\psi^*(\lambda z) = |\lambda|\psi^*(z)$, $\lambda \in \mathbb{C}$, $z \in \mathbb{C}^N$). Therefore, the domain of convergence $\mathcal{D}$ is either empty, or it is a *balanced* (i.e. $\lambda z \in \mathcal{D}$ for all $\lambda \in \mathbb{C}$ with $|\lambda| \le 1$ and $z \in \mathcal{D}$) domain of holomorphy. Every balanced domain of holomorphy is a domain of convergence of a series (1.1).

For every balanced domain $D$ in $\mathbb{C}^N$ there is a unique nonnegative function $h$ (so-called *Minkowski functional of* $D$) such that $h(\lambda z) = |\lambda|h(z)$ for all $\lambda \in \mathbb{C}$ and $z \in \mathbb{C}^N$, and $D = \{z \in \mathbb{C}^N;\ h(z) < 1\}$. In particular, if $\mathcal{D}$ is a domain of convergence of (1.1), then $h(z) \equiv \psi^*(z)$.

It is known that a balanced domain in $\mathbb{C}^N$ is a domain of holomorphy if and only if its Minkowski functional $h$ is an absolutely homogeneous plurisubharmonic function.

The number

$$\rho := 1/\limsup_{j\to\infty} \sqrt[j]{\|Q_j\|_{\mathbb{B}}}, \tag{1.4}$$

where $\mathbb{B} := \{z \in \mathbb{C}^N;\ \|z\| \le 1\}$, is called a *radius of convergence* of series (1.1) (with respect to a given norm $\|\cdot\|$).

If $N = 1$, then $\psi(z) = \frac{|z|}{\rho}$ and $\mathcal{D} = \rho\mathbb{B}$. If $N \ge 2$, then $\rho\mathbb{B} \subset \mathcal{D}$ but, in general, $\mathcal{D} \ne \rho\mathbb{B}$.

Series (1.1) is *normally geometrically* convergent in $\mathcal{D}$, i.e.

$$\limsup_{j\to\infty} \sqrt[j]{\|Q_j\|_K} < 1, \quad \limsup_{n\to\infty} \sqrt[n]{\|f - s_n\|_K} < 1, \tag{1.5}$$

for all compact sets $K \subset \mathcal{D}$, where $s_n := Q_o + \cdots + Q_n$ is the $n$th *partial sum of* (1.1).

**Definition 1.1.** We say that a function $f$ holomorphic in a neighborhood of a point $z^o \in \mathbb{C}^N$ possesses at the point $z^o$ *Ostrowski's gaps* $(m_k, n_k]$, if

$1^o$. $m_k$, $n_k$ are natural numbers such that $m_k < n_k < m_{k+1}$ $(k \ge 1)$, $\frac{n_k}{m_k} \to \infty$ as $k \to \infty$;

$2^o$. $\lim_{j\to\infty,\, j\in I} \sqrt[j]{\|Q_j\|_{\mathbb{B}}} = 0$, where $\mathbb{B}$ is the unit ball in $\mathbb{C}^N$,

$$Q_j(z) \equiv Q_j^{(f,z^o)}(z) := \sum_{|\alpha|=j} \frac{f^{(\alpha)}(z^o)}{\alpha!} z^\alpha = \frac{1}{j!}\left(\frac{d}{d\lambda}\right)^j f(z^o + \lambda z)_{|\lambda=0},$$

and $I := \bigcup_{k=1}^\infty (m_k, n_k]$, $(m_k, n_k]$ denoting the set of integers $j$ with $m_k < j \le n_k$.

Observe that $f_o(z) := \sum_{j\in I} Q_j(z - z^o)$ is an entire function such that the function $g := f - f_o$ possesses Ostrowski's gaps $(m_k, n_k]$ at $z^o$ with $Q_j^{(g,z^o)} = 0$ for $m_k < j \le n_k$, $k \ge 1$. Hence, a holomorphic function $f$ possesses Ostrowski's gaps $(m_k, n_k]$ at a point $z^o$ if and only if there exists an entire function $f_o$ such that $Q_j^{(f-f_o,z^o)} = 0$ for $m_k < j \le n_k$, $k \ge 1$.

Moreover, the maximal domain of existence $G = G_f$ of $f$ is identical with the maximal domain of existence of $f - f_o$.

**Definition 1.2.** We say that a function $f$ holomorphic in a neighborhood of a point $z^o$ possesses *Ostrowski's gaps relative to a sequence of positive integers* $\{n_k\}$, if $\{n_k\}$ is increasing and there exists a sequence of positive real numbers $\{q_k\}$ such that $q_k \to 0$ as $k \to \infty$ and $\lim_{j\to\infty, j\in I} \sqrt[j]{\|Q_j\|_{\mathbb{B}}} = 0$, where $I := \bigcup_{k=1}^{\infty}(\lfloor q_k n_k \rfloor, n_k]$.

A function $f$ possesses Ostrowski's gaps according to Definition 1.1 if and only if $f$ possesses Ostrowski's gaps according to Definition 1.2.

Indeed, if the conditions of Definition 1.1 are satisfied, then it is sufficient to put $q_k := m_k/n_k$.

If the conditions of Definition 1.2 are satisfied, consider two cases. If $m := \liminf_{k\to\infty} q_k n_k$ is finite, then the function $f$ is entire, so that $f$ has Ostrowski's gaps $(m_k, n_k]$ according to Definition 1 for any sequence $m_k, n_k$ satisfying $1^o$.

If $\liminf_{k\to\infty} q_k n_k = \infty$, then $f$ possesses Ostrowski's gaps $(\lfloor q_{k_l} n_{k_l} \rfloor, n_{k_l}]$ for a suitable chosen increasing subsequence $k_l$ of positive integers.

We say that a compact subset $K$ of $\mathbb{C}^N$ is *polynomially convex* if $K$ is identical with its *polynomially convex hull* $\hat{K} := \{a \in \mathbb{C}^N;\ |P(a)| \le \|P\|_K$ for every polynomial $P$ of $N$ complex variables$\}$. We say that an open set $\Omega$ in $\mathbb{C}^N$ is *polynomially convex*, if for every compact subset $K$ of $\Omega$ the polynomially convex hull $\hat{K}$ of $K$ is contained in $\Omega$.

The following theorem is known (see [7]). It is a multidimensional version of the classical Ostrowski's Theorem (see Theorem 3.1.1 in [1]).

**Theorem 1.** *If a holomorphic function $f$ possesses Ostrowski's gaps $(m_k, n_k]$ at a point $z^o \in \mathbb{C}^N$, then the maximal domain of existence $G = G_f$ of $f$ is one-sheeted and polynomially convex. Moreover, for every compact subset $K$ of $G$ we have*

$$\limsup_{k\to\infty} \|f - s_{n_k}\|_K^{1/n_k} < 1, \tag{1.6}$$

*where*

$$s_n(z) \equiv s_n^{(f,z^o)}(z) = \sum_{j=0}^{n} Q_j^{(f,z^o)}(z - z^o)$$

*is the nth partial sum of the Taylor series development of $f$ around $z^o$.*

**Corollary 1.1.** *If*

$$f(z^o + z) = \sum_{k=1}^{\infty} Q_{m_k}^{(f,z_o)}(z),$$

*where $m_k/m_{k+1} \to 0$ as $k \to \infty$, then $Q_j^{(f,z^o)} = 0$ for $j \notin \{m_k\}$ so that $f$ has Ostrowski's gaps $(m_k, n_k]$ with $n_k := m_{k+1} - 1$. Therefore, the maximal* *domain of existence $G_f$ of $f$ is identical with the domain of convergence $\mathcal{D}_f$ of the Taylor series development of $f$ around $z^o$, i.e.*

$$G_f = \mathcal{D}_f := \left\{z \in \mathbb{C}^N : \psi^*(z - z^o) < 1\right\},$$

*where* $\psi(z) := \limsup_{k\to\infty} \sqrt[m_k]{|Q^{(f,z^o)}_{m_k}(z)|}$.

The following result gives an $N$-dimensional version of W. Luh's Theorem 1 in [4]. In particular, it says that if a function $f$ holomorphic in a domain $G$ in $\mathbb{C}^N$ possesses Ostrowski's gaps at some point $z^o \in G$, then $f$ possesses the same property at every other point $a$ of the maximal domain of existence of $f$.

**Theorem 2.** *Let $f$ possess Ostrowski's gaps $(m_k, n_k]$ at a point $z^o \in \mathbb{C}^N$. Then*

$1^o$. *$f$ possesses Ostrowski's gaps $\left(m_{k_l}, \left[\frac{n_{k_l}}{l}\right]\right]$ at every point $a \in G_f$, where the sequence of natural numbers $\{k_l\}$ (independent of $a$) is chosen in such a way that $n_{k_l} \geq m_{k_l} l^2$ and $\left[\frac{n_{k_l}}{l}\right] < m_{k_{l+1}}$ for $l \geq 1$;*

$2^o$. *If $Q^{(f,z^o)}_j = 0$ for $m_k < j \leq n_k$, $k \geq 1$*[1]*, then the sequence $\left\{s^{(f,z^o)}_{m_k} - s^{(f,a)}_{m_k}\right\}$ converges to zero normally with order $n_k$ on $\mathbb{C}^N$, i.e.*

$$\limsup_{k\to\infty} \left\| s^{(f,z^o)}_{m_k} - s^{(f,a)}_{m_k} \right\|_K^{1/n_k} < 1$$

*for every compact set $K \subset \mathbb{C}^N$.*

By $2^o$ and Theorem 1 we get the following:

**Corollary 1.2.** *If $f$ possesses ordinary Ostrowski's gaps $(m_k, n_k]$ at a fixed point $z^o \in G$, then*

$$\limsup_{k\to\infty} \sqrt[n_k]{\left\| f - s^{(f,a)}_{m_k} \right\|_K} < 1$$

*for every point $a \in G_f$ and every compact subset $K$ of $G_f$.*

**Proof of Theorem 2.** $1^o$. Without loss of generality we may assume that $z^o = 0$ and

$$Q^{(f,z^o)}_j = 0, \quad m_k < j \leq n_k, \quad k \geq 1.$$

Given a fixed point $a \in G_f$, we have

$$Q^{(f,a)}_j(z) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(a + \lambda z) - s_{n_k}(a + \lambda z)}{\lambda^{j+1}} d\lambda,$$

[1]In such a case we say that $f$ possesses *ordinary* Ostrowski's gaps at $z^o$

$\|z\| \le 1$, $j > m_k$, $k \ge 1$, where $s_{n_k} = s_{n_k}^{(f,z_o)}$ (Observe that $s_{n_k}$ is a polynomial of degree at most $m_k$), and $0 < r < \min(\operatorname{dist}(a, \partial G_f), \operatorname{dist}(z^o, \partial G_f))$. By Theorem 1 there exist $M > 1$ and $0 < \theta < 1$ such that

$$\|f - s_{n_k}\|_{\mathbb{B}(a,r)} \le M\theta^{n_k}, \quad k \ge 1. \tag{1.7}$$

Therefore, by Cauchy inequalities,

$$\left\|Q_j^{(f,a)}\right\|_{\mathbb{B}} \le \frac{M}{r^j}\theta^{n_k}, \quad j > m_k, \quad k \ge 1. \tag{1.8}$$

Let $\{k_l\}$ be an increasing sequence of natural numbers such that

$$m_{k_{l+1}} > \left\lceil \frac{n_{k_l}}{l} \right\rceil, \quad \frac{n_{k_l}}{m_{k_l}} \ge l^2, \quad l \ge 1.$$

By (1.8) we get

$$\left\|Q_j^{(f,a)}\right\|_{\mathbb{B}}^{1/j} \le \frac{M}{r}\theta^{n_{k_l}/j} \le \frac{M}{l}\theta^l, \quad m_{k_l} < j \le \left\lceil \frac{n_{k_l}}{l} \right\rceil, \quad l \ge 1.$$

The choice of the sequence $\{k_l\}$ does not depend on $a \in G_f$. Therefore, $f$ possesses Ostrowski's gaps $\left(m_{k_l}, \left\lceil \frac{n_{k_l}}{l} \right\rceil\right]$ at every point $a$ of $G_f$ (according to Definition 1.1). The proof of the case $1^o$ is ended.

$2^o$. Observe that for $\|z - a\| \le \frac{1}{2}r$ we have

$$\left|f(z) - s_{m_k}^{(f,a)}(z)\right| = \sum_{m_k+1}^{\infty} \left|Q_j^{(f,a)}(z-a)\right| \le \sum_{m_k+1}^{\infty} \left\|Q_j^{(f,a)}\right\|_{\mathbb{B}} \left(\frac{r}{2}\right)^j,$$

which by (1.8) gives

$$\left|f(z) - s_{m_k}^{(f,a)}(z)\right| \le \sum_{p_k+1}^{\infty} 2^{-j} M\theta^{n_k} \le M\theta^{n_k}, \quad k \ge 1, \quad \|z - a\| \le \frac{r}{2}. \tag{1.9}$$

By (1.7) and (1.9) we get

$$\left\|s_{m_k}^{(f,z^o)} - s_{m_k}^{(f,a)}\right\|_{\mathbb{B}(a,\frac{1}{2}r)} \le 2M\theta^{n_k}, \quad k \ge 1. \tag{1.10}$$

Observe that for $z \in \mathbb{C}^N$

$$\begin{aligned}\left|s_n^{(f,z^o)}(z)\right| &\le \sum_{j=0}^{n} \left\|Q_j^{(f,z^o)}\right\|_{\mathbb{B}} \|z - z^o\|^j \le \sum_{0}^{n} \frac{\|f\|_{\mathbb{B}(z^o,r)}}{r^j} \|z - z^o\|^j \\ &\le (n+1)\|f\|_{\mathbb{B}(z^o,r)} \left(1 + \frac{\|z\| + \|z_o\|}{r}\right)^n.\end{aligned}$$

Put $M := \|f\|_{\mathbb{B}(z^o,r) \cup \mathbb{B}(a,r)}$ and $c := \max\{\|z^o\|, \|a\|\}$. Then for $z \in \mathbb{C}^N$

$$\begin{aligned}u_k(z) &:= \frac{1}{n_k} \log \left|s_{m_k}^{(f,z^o)}(z) - s_{m_k}^{(f,a)}(z)\right| \\ &\le \frac{1}{n_k} \log[2M(m_k+1)] + \frac{m_k}{n_k} \log\left(1 + \frac{\|z\| + \|c\|}{r}\right).\end{aligned}$$

It follows that the sequence of plurisubharmonic functions $\{u_k\}$ is locally uniformly upper bounded in $\mathbb{C}^N$, and

$$u(z) := \limsup_{k\to\infty} u_k(z) \le 0, \quad z \in \mathbb{C}^n.$$

Therefore, the plurisubhamonic function $u^* = \mathit{const.}$

By (1.10) $u_k(z) \le \frac{1}{n_k}\log 2M + \log\theta$ for $z \in \mathbb{B}(a,r)$, $k \ge 1$. Hence $u^* \le \log\theta$ in $\mathbb{C}^N$ which ends the proof of $2^o$. □

**2. E. Fabry's Theorem.** Now we shall present a multidimensional version of E. Fabry's Theorem (Theorem 2.2.1 in [1]). Let $f$ be a function of N complex variables holomorphic in a neighborhood of 0 with a gap Taylor series development

$$f(z) = \sum_{k=1}^{\infty} Q_{m_k}(z), \quad m_k < m_{k+1}. \tag{2.1}$$

Put $\psi(z) := \limsup_{k\to\infty} \sqrt[m_k]{|Q_{m_k}(z)|}$, $h(z) := \psi^*(z)$. It is known that $\mathcal{D} := \{z \in \mathbb{C}^N;\, h(z) < 1\} = \{a \in \mathbb{C}^N;$ series (2.1) is convergent in a neighborhood of $a\}$ is a domain of convergence of (2.1).

**Theorem 3.** *If* $\lim_{k\to\infty}\frac{k}{m_k} = 0$, *then the domain of convergence* $\mathcal{D}$ *of the series* (2.1) *is identical with the maximal domain of existence* $G_f$ *of* $f$.

**Proof.** Without loss of generality we may assume that $\mathcal{D} \ne \mathbb{C}^N$.

Due to Fabry we know that Theorem 3 is true for $N = 1$. It is also well known (by Bedford–Taylor Theorem on negligible sets) that the set $E := \{z \in \mathbb{C}^N;\, \psi(z) < \psi^*(z)\}$ is pluripolar. Therefore, in particular, the set $E$ is of $2N$-dimensional Lebesgue measure zero.

Suppose Theorem 3 is not true for some $N > 1$. Then there is a function $g$ holomorphic in a ball $B(z_o, R)$ with $z_o \in \mathcal{D}$, $R > r := \operatorname{dist}(z_o, \partial\mathcal{D})$ such that $g(z) = f(z)$ for $z \in B(z_o, r)$.

Let $b_o$ be a fixed point of $\partial\mathcal{D}$ such that $\|b_o - z_o\| = r$.

Since the ball $B(z_o, r)$ is *non-thin* at the point $b_o$,we have

$$\limsup_{z\to b_o, z\in B(z_o,r)} \psi^*(z) = \psi^*(b_o).$$

Therefore, there is a sequence $\{z'_k\} \subset B(z_o, r)$ such that $z'_k \to b_o$, and $\psi^*(z'_k) \to \psi^*(b_o)$ as $k \to \infty$. It follows that $\psi^*(b_o) \le 1$. Since $b_o \in \partial D$, we have $\psi^*(b_o) \ge 1$. Therefore, $\psi^*(b_o) = 1$.

We know that the $2N$-dimensional Lebesgue measure $v_{2N}(E) = 0$. Therefore, by the sub-mean-value property, for every $k \ge 1$ there is a point $z_k \in B(z'_k, \frac{1}{k}) \cap B(z_o, r) \setminus E$ such that $\psi(z_k) = \psi^*(z_k)$, $|\psi^*(z'_k) - \psi(z_k)| < \frac{1}{k}$. It is clear that the sequence $\{z_k\}$ satisfies the following properties:

$$z_k \in B(z_o, r),\ z_k \to b_o,\ \psi(z_k) = \psi^*(z_k),\ \psi(z_k) \to \psi^*(b_o).$$

Put $b_k = z_k/\psi(z_k)$ $(k \geq 1)$. Then $\psi(b_k) = \psi^*(b_k) = 1$, in particular, $b_k \in \partial\mathcal{D}$ for $k \geq 1$, and $b_k \to b_o$ as $k \to \infty$.

Fix $k$ so large that $b := b_k \in B(z_o, R)$. Put

$$G_r := \{\lambda \in \mathbb{C};\ \lambda b \in B(z_o, r)\},$$

$$G_R := \{\lambda \in \mathbb{C};\ \lambda b \in B(z_o, R)\}.$$

One can easily check that the sets $G_r$, $G_R$ are open, convex, nonempty (because $\lambda_o b \in G_r$ for $\lambda_o := \psi(z_k)$, and $G_r \subset G_R$). Moreover, $G_r \subset \Delta := \{|\lambda| < 1\}$, and $1 \in G_R$.

The function $f(\lambda b)$ (resp., $g(\lambda b)$) is holomorphic in $\Delta$ (resp., in $G_R$), and $f(\lambda b) = g(\lambda b)$ for $\lambda \in G_r$. Therefore, $f(\lambda b) = g(\lambda b)$ on $\Delta \cap G_R$. It follows that $g(\lambda b)$ is an analytic continuation of $f(\lambda b)$ across $\lambda = 1$, contrary to the Fabry Theorem for $N = 1$. We have got a contradiction showing that Theorem 3 is true. $\square$

**Remark.** The present proof of Theorem 3 – with no assumption on the continuity of the function $\psi^*$ – is a joint result of the author and Professor Azimbay Sadullaev.

**3. Fatou–Hurwitz–Polya Theorem.** First we shall state Fatou–Hurwitz –Polya Theorem for a series of homogeneous polynomials of $N$ complex variables.

**Theorem 4.** *Let $f$ be a function holomorphic in a neighborhood of $0 \in \mathbb{C}^N$. Let*

$$f(z) = \sum_{0}^{\infty} Q_j(z), \quad Q_j(z) = \sum_{|\alpha|=j} \frac{f^{(\alpha)}(0)}{\alpha!} z^\alpha, \tag{3.0}$$

*be its Taylor series development around $0$. Then there exists a sequence $\epsilon = \{\epsilon_j\}$ with $\epsilon_j \in \{-1, 1\}$ (resp., $\epsilon_j \in \{0, 1\}$) such that the function*

$$f_\epsilon(z) := \sum_{j=0}^{\infty} \epsilon_j Q_j(z), \quad z \in \mathcal{D},$$

*has no analytic continuation across any boundary point of the domain of convergence $\mathcal{D} := \{\psi^*(z) < 1\}$ of series (3.0), where*

$$\psi(z) := \limsup_{j \to \infty} \sqrt[j]{|Q_j(z)|}.$$

For $N = 1$ this theorem (with $\epsilon_j \in \{-1, 1\}$) is due to Fatou–Hurwitz–Polya (Theorem 4.2.8 in [1]).

Now, we shall present an $N$-dimensional version of the Fatou–Hurwitz–Polya theorem for *$N$-tuple power series*

$$f(z) = \sum_{|\alpha| \geq 0} c_\alpha z^\alpha, \tag{3.1}$$

where $c_\alpha z^\alpha$ is a monomial of $N$ complex variables $z = (z_1, \dots, z_N)$ of degree $|\alpha| := \alpha_1 + \dots + \alpha_N$. The set $\mathcal{D} := \{a \in \mathbb{C}^N$; the series (3.1) is absolutely convergent in a neighborhood of $a\}$ is called a *domain of convergence* of the multiple power series (3.1).

It is known that $\mathcal{D} = \{z \in \mathbb{C}^N;\ h(z) < 1\}$ is a complete $N$-circular (hence, in particular, $\mathcal{D}$ is balanced) domain whose Minkowski's functional $h \equiv h_{\mathcal{D}}$ is given by the formula $h(z) = M^*(z)$, where

$$\begin{aligned} M(z) &:= \limsup_{|\alpha|\to\infty} \sqrt[|\alpha|]{|c_\alpha z^\alpha|} \\ &= \limsup_{k\to\infty} \max\left\{ \sqrt[|\alpha|]{|c_\alpha z^\alpha|};\ |\alpha| = k \right\}, \quad z \in \mathbb{C}^N. \end{aligned} \tag{3.2}$$

Moreover, $h(z_1, \dots, z_N) = h(|z_1|, \dots, |z_N|)$ for all $z \in \mathbb{C}^N$, and $h$ is continuous (see [2], Lemma 1.7.1 (b)).

**Theorem 5.** *If the domain of convergence $\mathcal{D}$ of* (3.1) *is not empty, then there exists a multiple sequence $\epsilon = \{\epsilon_\alpha\}$ with $\epsilon_\alpha \in \{-1, 1\}$ (resp., with $\epsilon_\alpha \in \{0, 1\}$) such that the function*

$$f_\epsilon(z) := \sum_{|\alpha|\ge 0} \epsilon_\alpha c_\alpha z^\alpha, \quad z \in \mathcal{D},$$

*has no analytic continuation across any boundary point of $\mathcal{D}$.*

We shall see that Theorems 4 and 5 are direct consequences of the following Lemma 3.2.

Let $\mathcal{X} := \{0, 1\}^{\mathbb{N}}$ (resp. $\{-1, 1\}^{\mathbb{N}}$) be the space of all sequences $x = (x_1, x_2, \dots)$ where $x_j = 0$, or $x_j = 1$ (resp. $x_j = -1$, or $x_j = 1$) for $j = 1, 2, \dots$. Endow $\mathcal{X}$ in the topology determined by the metric

$$\rho(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^j} \frac{|x - y|_j}{1 + |x - y|_j},$$

where

$$|x - y|_j := \max\{|x_k - y_k|;\ k = 1, \dots, j\}.$$

One can easily check that $\mathcal{X}$ is a complete metric space, and therefore, it has Baire property.

Moreover, in the topology a sequence $\{x(n)\}$ of elements of $\mathcal{X}$ converges to an element $x \in \mathcal{X}$ if and only if for every $k_o \in \mathbb{N}$ there exists $n_o \in \mathbb{N}$ such that $x_k(n) = x_k$ for $k = 1, \dots, k_o$, $n \ge n_o$.

**Remark 3.1.** Let $\{f_k\}$ be a sequence of holomorphic functions in an open subset $\Omega$ of $\mathbb{C}^n$. Then the following three conditions are equivalent:

(1) the series $\sum_1^\infty |f_k(z)|$ converges at each point $z \in \Omega$, and its sum $\varphi(z) := \sum_1^\infty |f_k(z)|$ is locally bounded on $\Omega$;

(2) the series $\sum_1^\infty f_k$ converges locally normally in $\Omega$, i.e. for every point $a$ of $\Omega$ there exists a neighborhood $U$ of $a$ such that the series $\sum_1^\infty \|f_k\|_U$ is convergent;

(3) the series $\sum_1^\infty |f_k|$ converges locally uniformly in $\Omega$.

**Proof.** It is clear that (2) $\Rightarrow$ (3) $\Rightarrow$ (1).

Suppose now (1) is true, and let $E(a,r) := \{z \in \mathbb{C}^n;\ |z_j - a_j| < r\ (j = 1, \dots, n)\}$ be a polydisk whose closure is contained in $\Omega$. Then there is a positive constant $M$ such that $\sum_1^\infty |f_k(z)| \le M$ for all $z \in E(a,r)$. By the Cauchy integral formula

$$|f_k(z)| \le \mu_k := \left(\frac{1}{\pi r}\right)^n \int_0^{2\pi} \dots \int_0^{2\pi} |f_k(a_1 + re^{it_1}, \dots, a_n + re^{it_n})| dt_1 \dots dt_n,$$

for all $z \in E(a, \frac{r}{2})$ and $k \ge 1$.

By Lebesgue monotonous convergence theorem the series $\sum_1^\infty \mu_k$ is convergent, and so is the series $\sum_1^\infty \|f_k\|_U$ with $U := E(a, \frac{r}{2})$. □

We shall see that our extensions of the classical Fatou–Hurwitz–Polya Theorem (Theorem 4.2.8 in [1]) are a direct consequence of the following Lemma 3.2 (slight modification of Lemma 5, p. 97 in [5]).

**Lemma 3.2.** *Let $\mathcal{X}$ denote any of the two metric spaces $\{0,1\}^{\mathbb{N}}$ or $\{-1,1\}^{\mathbb{N}}$. Let $\{f_k\}$ be a sequence of holomorphic functions in an open neighborhood $\Omega$ of the closure of a ball $B = B(w,r)$ such that the series $\sum_1^\infty |f_k(z)|$ converges at every point $z \in B$. Let $a$ be a boundary point of $B$.*

*Then, either the series $\sum_1^\infty f_k$ is normally convergent on a neighborhood of $a$, or there exists a subset $\mathcal{R}$ of $\mathcal{X}$ of the first category such that for every $x \in \mathcal{X} \setminus \mathcal{R}$ the function $f_x(z) := \sum_k x_k f_k(z)$, $z \in B$, has a singular point at $a$ (in other words, $f_x$ cannot be analytically continued to any neighborhood of $a$).*

**Proof.** Given a natural number $m$, let $\mathcal{R}_m$ denote the set of all $x \in \mathcal{X}$ such that there exists a holomorphic function $\tilde{f}_x$ on $E_m$ (where $E_m$ is the polydisk $E_m := E\big(a, \frac{1}{m}\big)$ with center $a$ and radius $\frac{1}{m}$) such that $|\tilde{f}_x(z)| \le m$ on the polydisk, and $\tilde{f}_x(z) = f_x(z)$ for all $z \in B \cap E_m$. By definition, we put $\mathcal{R}_m = \emptyset$, if $m < 1/\operatorname{dist}(a, \partial\Omega)$.

It is clear that the set $\mathcal{R} := \bigcup_1^\infty \mathcal{R}_m \equiv \{x \in \mathcal{X};\ f_x$ has an analytic continuation across $a\}$.

The lemma will be proved if we show that the following two claims are true.

**Claim 1.** *The set $\mathcal{R}_m$ is closed in the space $\mathcal{X}$.*

**Claim 2.** *If the interior of $\mathcal{R}_m$ is not empty, then the series $\sum_1^\infty f_k$ is normally convergent on a neighborhood of $a$.*

Indeed, if the series $f_x := \sum_1^\infty x_k f_k$ converges normally on no neighborhood $U$ of $a$, then for every $m \geq 1$ the set $\mathcal{R}_m$ is closed and has empty interior. Hence, the set $\mathcal{R} := \bigcup_1^\infty \mathcal{R}_m \equiv \{x \in \mathcal{X};\ f_x$ has an analytic continuation $\tilde{f}_x$ across $a\}$ is of the first category, and for every $x \in \mathcal{X} \setminus \mathcal{R}$ the function $f_x$ has a singular point at $a$, i.e. $f_x$ has no analytic continuation across $a$. We say that a function $\tilde{f}_x$ holomorphic on a polydisk $E$ with center $a$ is an analytic continuation of $f_x$ across $a$, if $\tilde{f}_x(z) = f_x(z)$ on $B \cap E$.

**Proof of Claim 1.** Let $\{x(j)\}$ be a sequence of elements of $\mathcal{R}_m$ convergent to $x \in \mathcal{X}$. Let $\{h_j\} \equiv \{\tilde{f}_{x(j)}\}$ be a sequence of holomorphic functions on $E_m$ such that $|h_j(z)| \leq m$ on $E_m$ and $h_j(z) = f_{x(j)}(z)$ on the intersection $B \cap E_m$ for $j \geq 1$ . Observe that for every $k_o$ there exists $j_o$ such that $|f_{x(j)}(z) - f_x(z)| \leq \sum_{k>k_o} 2|f_k(z)|$ for all $z \in B \cap E_m$ and for all $j > j_o$. It follows that the sequence $\{h_j\}$ is convergent at each point of $B \cap E_m$. By Vitali's theorem the sequence $\{h_j\}$ is locally uniformly convergent on $E_m$ to a holomorphic function $h$ bounded by $m$ and identical with $f_x$ on $E_m \cap B$, which shows that $x \in \mathcal{R}_m$.

**Proof of Claim 2.** If $\mathcal{R}_m$ has a nonempty interior, then there exist $x(0) = (x_1(0), x_2(0), \dots) \in \mathcal{R}_m$ and a natural number $k_o$ such that

$$(*) \qquad x \in \mathcal{X},\ x_j = x_j(0) \quad (j = 1, \dots, k_o) \quad \Longrightarrow \quad x \in \mathcal{R}_m.$$

Put

$$M := \sup\left\{\sum_{k=1}^{k_0} |f_k(z)|;\ z \in E_m\right\}, \quad u_k := \Re f_k, \quad v_k := \Im f_k.$$

By implication (2) $\Rightarrow$ (3) of Remark 3.1 it is sufficient to show that

$$(**) \qquad \sum_{k=1}^{\infty} |f_k(z)| \leq M + 4m, \quad z \in E_m.$$

Let $A$ be a finite subset of $\mathbb{N} \setminus [1, k_0]$. Given a fixed point $z$ of $E_m$, put

$$A_1 := \{k \in A;\ u_k(z) \geq 0\}, \quad A_2 := \{k \in A;\ u_k(z) < 0\}.$$

It is clear that $A = A_1 \cup A_2$, $A_1 \cap A_2 = \emptyset$. Consider two cases.

**Case 1**: $\mathcal{X} = \{0,1\}^{\mathbb{N}}$. Let $x(j) = (x_1(j), x_2(j), \dots)$ $(j = 1, 2)$ be two points of the interior of $\mathcal{R}_m$ defined by the formulas:

$$\begin{aligned}
&x_k(j) = x_k(0), \quad k = 1, \dots, k_0, \quad j = 1, 2;\\
&x_k(j) = x_k(0), \quad k > k_0, \quad k \notin A, \quad j = 1, 2;\\
&x_k(1) = 1, \quad x_k(2) = 0, \quad k \in A_1;\\
&x_k(1) = 0, \quad x_k(2) = 1, \quad k \in A_2.
\end{aligned}$$

Then

$$\sum_{k\in A}|u_k(z)| \le \left|\sum_{k\in A}(x_k(1)-x_k(2))f_k(z)\right| = |\tilde{f}_{x(1)}(z)-\tilde{f}_{x(2)}(z)| \le 2m.$$

By the arbitrary property of $A$ and $z$ one gets

$$\sum_{k=k_0+1}^{\infty}|u_k(z)| \le 2m, \quad z\in E_m.$$

The same argument gives

$$\sum_{k=k_0+1}^{\infty}|v_k(z)| \le 2m, \quad z\in E_m.$$

Hence

$$\sum_{k=1}^{\infty}|f_k(z)| = \left(\sum_{k=1}^{k_0}+\sum_{k=k_0+1}^{\infty}\right)|f_k(z)| \le M+4m, \quad z\in E_m.$$

**Case 2:** $\mathcal{X}=\{-1,1\}^{\mathbb{N}}$. Now we define two elements $x(1)$, $x(2)$ of the interior of $\mathcal{R}_m$ by the formulas:

$$\begin{aligned}
&x_k(j)=x_k(0), \quad k=1,\dots,k_0, \quad j=1,2;\\
&x_k(j)=x_k(0), \quad k>k_0, \quad k\notin A, \quad j=1,2;\\
&x_k(1)=1, \quad x_k(2)=-1, \quad k\in A_1;\\
&x_k(1)=-1, \quad x_k(2)=1, \quad k\in A_2.
\end{aligned}$$

Then

$$2\sum_{k\in A}|u_k(z)| \le \left|\sum_{k\in A}\left(x_k(1)-x_2(2)\right)f_k(z)\right| = |\tilde{f}_{x(1)}(z)-\tilde{f}_{x_k(2)}(z)| \le 2m.$$

Hence, by the analogous argument as in the proof of the case 1, we get

$$\sum_{k=1}^{\infty}|f_k(z)| \le M+4m, \quad z\in E_m,$$

which ends the proof of the case 2. □

**Corollary 3.3.** *Let $\{f_k\}$ be a sequence of holomorphic functions on an open set $\Omega\subset\mathbb{C}^N$. Let $D$ denote the set of all points $a$ in $\Omega$ such that the series $\sum_1^\infty f_k$ is absolutely convergent at every point of a neighborhood of $a$. Assume that the sum $\varphi(z):=\sum_1^\infty|f_k(z)|$ is locally bounded in $D$, and $\bar{D}\subset\Omega$. Let $\mathcal{X}$ be any of the two metric spaces $\{0,1\}^{\mathbb{N}}$ or $\{-1,1\}^{\mathbb{N}}$.*

*Then there exists a subset $\mathcal{R}$ of $\mathcal{X}$ of the first category such that for every point $x\in\mathcal{X}\setminus\mathcal{R}$ the holomorphic function $f_x(z):=\sum_{k=1}^\infty x_kf_k(z)$, $z\in D$, cannot be continued analytically across any boundary point of $D$.*

**Proof.** Let $\{w_j\}$ be the sequence of all rational points of $D$ (or any countable dense subset of $D$). Let $a_j$ be a point of $\partial D$ such that $\|w_j - a_j\| = \operatorname{dist}(w_j, \partial D)$. By Lemma 3.2 for every $j$ there exists a subset $\mathcal{R}_j$ of $\mathcal{X}$ of the first category such that for every $x \in \mathcal{X} \setminus \mathcal{R}_j$ the function $f_x$ has a singular point at $a_j$. The set $\mathcal{R} \coloneqq \bigcup \mathcal{R}_j$ is again of the first category such that for every $x \in \mathcal{X} \setminus \mathcal{R}$ the function $f_x$ has analytic extension across no boundary point of $D$. □

**Proof of Theorems 4 and 5.** It is sufficient to apply Lemma 3.2 with $\Omega = \mathbb{C}^N$, with $f_k = Q_k$ and $f_k = c_{\alpha(k)} z^{\alpha(k)}$ $(k \in \mathbb{Z}_+)$, respectively, where $\alpha : \mathbb{Z}_+ \ni k \mapsto \alpha(k) \in \mathbb{Z}_+^N$ is a one-to-one mapping, and with $D$ replaced by the domain of convergence $\mathcal{D}$ of the corresponding power series. □

**Remark 3.4.** The author would like to draw reader's attention to the fact that, unfortunately, the proofs of Theorems 4 and 5 published in [6] contain flaws.

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